

Elastic axial extension of a prismatic rod

This text is integrating part of the homonymous link in [PEEI: a computer program for the numerical solution of systems of partial differential equations](#).

System of measurement: International System of Units, with the exception of the force that is expressed in $\text{N} \times 10^{-12}$.

Coordinate system: Cartesian

Coordinates: \underline{x} of which: $\underline{x} \equiv \{x_i; i=1,3\}$ $[x_i] = [\text{length}]$ $\mathbb{R}(\underline{x}_i) \equiv (-\infty, \infty)$, \underline{x} a point of the deformed medium.

Coordinate versors: $\{\mathbf{v}_i; i=1,3\}$

Unknown functions: $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \tau_{11}, \tau_{12}, \tau_{13}, \tau_{22}, \tau_{23}, \tau_{33}\}$ of which: $\mathbf{s}_i = \mathbf{x}_i - \mathbf{X}_i$, $[\mathbf{s}_i] = [\text{length}]$, $\underline{X} \equiv \{X_i; i=1,3\}$, \underline{X} the position of the point \underline{x} in the undeformed medium, $\mathbf{s} \equiv \sum_{i=1,3} (\mathbf{s}_i \cdot \mathbf{v}_i)$, \mathbf{s} the displacement of the point \underline{X} , $\{\tau_{11}, \tau_{12}, \tau_{13}, \tau_{22}, \tau_{23}, \tau_{33}\}$ the six independent components of the stress tensor, $[\tau_{ij}] = [\text{stress}]$, $\tau_{ij} = \tau_{ji}$.

Differential analytical model:

$$\partial \tau_{11}(\underline{x}) / \partial x_1 + \partial \tau_{12}(\underline{x}) / \partial x_2 + \partial \tau_{13}(\underline{x}) / \partial x_3 + F_1(\underline{x}) = 0$$

$$\partial \tau_{12}(\underline{x}) / \partial x_1 + \partial \tau_{22}(\underline{x}) / \partial x_2 + \partial \tau_{23}(\underline{x}) / \partial x_3 + F_2(\underline{x}) = 0$$

$$\partial \tau_{13}(\underline{x}) / \partial x_1 + \partial \tau_{23}(\underline{x}) / \partial x_2 + \partial \tau_{33}(\underline{x}) / \partial x_3 + F_3(\underline{x}) = 0$$

$$\{(1+\nu) \cdot \tau_{ij}(\underline{x}) - \delta_{ij} \cdot \nu \cdot (\tau_{11}(\underline{x}) + \tau_{22}(\underline{x}) + \tau_{33}(\underline{x})) - E \cdot (\partial \mathbf{s}_i(\underline{x}) / \partial x_j + \partial \mathbf{s}_j(\underline{x}) / \partial x_i) / 2 = 0; j=i,3; i=1,3\}$$

of which: $\mathbf{F} \equiv \sum_{i=1,3} (F_i \cdot \mathbf{v}_i)$, \mathbf{F} the body force per unit volume, $\{\delta_{ij}=0; \forall i \neq j\}$ $\{\delta_{ij}=1; \forall i=j\}$, E Young's modulus, ν Poisson's ratio, $E=0.21$ $\nu=0.3$.

Related relations:

$$\varepsilon_{ij}(\underline{x}) = (\partial \mathbf{s}_i(\underline{x}) / \partial x_j + \partial \mathbf{s}_j(\underline{x}) / \partial x_i) / 2 = (1+\nu) \cdot \tau_{ij}(\underline{x}) / E - \delta_{ij} \cdot \nu \cdot (\tau_{11}(\underline{x}) + \tau_{22}(\underline{x}) + \tau_{33}(\underline{x})) / E \quad (1)$$

$$\omega_{ij}(\underline{x}) = (\partial \mathbf{s}_i(\underline{x}) / \partial x_j - \partial \mathbf{s}_j(\underline{x}) / \partial x_i) / 2 \quad (2)$$

$$\mathbf{T}_i(\underline{x}) = \sum_{j=1,3} (\tau_{ji}(\underline{x}) \cdot \mathbf{n}_j(\underline{x})) \quad (3)$$

$$\mathbf{s}_i(\underline{x}_B) = \mathbf{s}_i(\underline{x}_A) + \sum_{j=1,3} (\omega_{ij}(\underline{x}_A) \cdot (\mathbf{x}_{Bj} - \mathbf{x}_{Aj})) + \int_{A,B} (\Theta_i(\mathbf{c}) \cdot d\mathbf{c}) \quad (4)$$

$$\Theta_i(\mathbf{c}) \equiv \sum_{j=1,3} (\varepsilon_{ij}(\underline{x}(\mathbf{c})) \cdot \mathbf{x}_j'(\mathbf{c}) + (\mathbf{x}_{Bj} - \mathbf{x}_j(\mathbf{c})) \cdot \sum_{k=1,3} ((\partial \varepsilon_{ik}(\underline{x}(\mathbf{c})) / \partial x_j - \partial \varepsilon_{jk}(\underline{x}(\mathbf{c})) / \partial x_i) \cdot \mathbf{x}_k'(\mathbf{c}))) \quad (5)$$

of which [here](#), $\varepsilon_{ij} = \varepsilon_{ji}$, $\mathbf{T}(\underline{x}) \equiv \sum_{i=1,3} (\mathbf{T}_i(\underline{x}) \cdot \mathbf{v}_i)$ $\mathbf{n}(\underline{x}) \equiv \sum_{i=1,3} (\mathbf{n}_i(\underline{x}) \cdot \mathbf{v}_i)$, $\mathbf{T}(\underline{x})$ the stress vector in a point of a plane with normal outward versor $\mathbf{n}(\underline{x})$, $\underline{x}(\mathbf{c}) \equiv \{x_i(\mathbf{c}); i=1,3\}$ $\underline{x}_A \equiv \{x_{Ai}; i=1,3\} = \underline{x}(A)$ $\underline{x}_B \equiv \{x_{Bi}; i=1,3\} = \underline{x}(B)$

Definition set: $\{\underline{x}/0 \leq x_1 \leq L_1; 0 \leq x_2 \leq L_2; 0 \leq x_3 \leq L_3\}$ $L_1=1$ $L_2=2$ $L_3=10$.

Conditions:

$$F_1(\underline{x})=F_2(\underline{x})=0 \quad \{S_i(\underline{x}_A)=0; i=1,3\} \quad \partial S_1(L_1/2, x_2, L_3)/\partial x_2 = \partial S_1(\underline{x}_P)/\partial x_3 = \partial S_2(\underline{x}_P)/\partial x_3 = 0 \quad (6)$$

where $\underline{x}_A \equiv \{L_1/2, L_2/2, L_3\}$ $\underline{x}_P \equiv \{L_1/2, L_2/2, x_3\}$.

$$\begin{aligned} T_1(x_1, x_2, 0) &= T_2(x_1, x_2, 0) = T_1(0, x_2, x_3) = T_2(0, x_2, x_3) = T_3(0, x_2, x_3) = T_1(L_1, x_2, x_3) = T_2(L_1, x_2, x_3) = \\ T_3(L_1, x_2, x_3) &= T_1(x_1, 0, x_3) = T_2(x_1, 0, x_3) = T_3(x_1, 0, x_3) = T_1(x_1, L_2, x_3) = T_2(x_1, L_2, x_3) = T_3(x_1, L_2, x_3) = 0 \end{aligned}$$

From these and (3) follows

$$\begin{aligned} \tau_{13}(x_1, x_2, 0) &= \tau_{23}(x_1, x_2, 0) = \tau_{11}(0, x_2, x_3) = \tau_{12}(0, x_2, x_3) = \tau_{13}(0, x_2, x_3) = \tau_{11}(L_1, x_2, x_3) = \tau_{12}(L_1, x_2, x_3) = \\ \tau_{13}(L_1, x_2, x_3) &= \tau_{12}(x_1, 0, x_3) = \tau_{22}(x_1, 0, x_3) = \tau_{23}(x_1, 0, x_3) = \tau_{12}(x_1, L_2, x_3) = \tau_{22}(x_1, L_2, x_3) = \tau_{23}(x_1, L_2, x_3) = 0 \end{aligned}$$

Related files: [mad.txt](#)

CASE 1

Conditions: $F_3(\underline{x})=0$ $T_3(x_1, x_2, 0)=-P=-0.05$. This and (3) imply $\tau_{33}(x_1, x_2, 0)=P$.

Exact solution:

From previous conditions follows $\tau_{11}(\underline{x})=\tau_{12}(\underline{x})=\tau_{13}(\underline{x})=\tau_{22}(\underline{x})=\tau_{23}(\underline{x})=0$ $\tau_{33}(\underline{x})=P$. These and (1) imply

$$\varepsilon_{12}(\underline{x})=\varepsilon_{13}(\underline{x})=\varepsilon_{23}(\underline{x})=0 \quad \varepsilon_{11}(\underline{x})=\varepsilon_{22}(\underline{x})=-v \cdot P/E \quad \varepsilon_{33}(\underline{x})=P/E \quad (7)$$

From these and (6) follows $\partial S_2(\underline{x}_A)/\partial x_1 = \partial S_3(\underline{x}_A)/\partial x_1 = \partial S_3(\underline{x}_A)/\partial x_2 = \partial S_1(\underline{x}_A)/\partial x_3 = \partial S_2(\underline{x}_A)/\partial x_3 = \partial S_1(\underline{x}_A)/\partial x_2 = 0$ and then (for (2)) $\omega_{ij}(\underline{x}_A)=0$. This, $\{S_i(\underline{x}_A)=0; i=1,3\}$ and (4) imply

$$S_i(\underline{x}_B) = \int_{A,B} (\Theta_i(c) \cdot dc) \quad (8)$$

of which $\underline{x}(A) \equiv \underline{x}_A$. From (7) and (5) follows

$$\Theta_i(c) = \sum_{j=1,3} (\varepsilon_{ij}(\underline{x}(c)) \cdot x_j'(c)) \quad (9)$$

Are placed

$$\begin{aligned} \int_{A,B} (\Theta_i(c) \cdot dc) &= \int_{A,P} (\Theta_i(c) \cdot dc) + \int_{P,Q} (\Theta_i(c) \cdot dc) + \int_{Q,B} (\Theta_i(c) \cdot dc) \quad \underline{x}(P) \equiv \underline{x}_P \quad \underline{x}(Q) \equiv \{L_1/2, x_2, x_3\} \quad \underline{x}(B) \equiv \{x_1, x_2, x_3\} \\ \{x_1'(c) &= x_2'(c) = 0, x_3'(c) = -1; \forall c \in [A, P]\} \quad \{x_1'(c) = x_3'(c) = 0, x_2'(c) = 1; \forall c \in [P, Q]\} \quad \{x_2'(c) = x_3'(c) = 0, \\ x_1'(c) &= 1; \forall c \in [Q, B]\} \end{aligned} \quad (10)$$

These, (9) and (8) imply $S_i(\underline{x}_B) = \int_{A,P} (\varepsilon_{i3}(\underline{x}(c)) \cdot dc) + \int_{P,Q} (\varepsilon_{i2}(\underline{x}(c)) \cdot dc) + \int_{Q,B} (\varepsilon_{i1}(\underline{x}(c)) \cdot dc)$. This and (7) imply

$$S_1(\underline{x}) = v \cdot P \cdot (L_1/2 - x_1)/E \quad S_2(\underline{x}_B) = v \cdot P \cdot (L_2/2 - x_2)/E \quad S_3(\underline{x}) = P \cdot (x_3 - L_3)/E \quad (11)$$

Note: In the following diagrams of this CASE 1, the symbols + (plus), □ (empty square) and ■ (full square) are respectively inherent to \underline{x} , \underline{X} determined by means of $X_i = x_i - S_i$ and (11), and \underline{x} determined by means of $X_i = x_i - S_i$ where S_i is calculated by PEEI.

Case 1-3-3-3: [points-3-3-3.txt](#), [mem-3-3-3.bin](#), [cond-1-3-3-3.txt](#), [sol-1-3-3-3.txt](#), [plot-1-3-3-3-1.jpg](#), [plot-1-3-3-3-2.jpg](#), [plot-1-3-3-3-3.jpg](#)

Case 1-3-3-5: [points-3-3-5.txt](#), [mem-3-3-5.bin](#), [cond-1-3-3-5.txt](#), [sol-1-3-3-5.txt](#), [plot-1-3-3-5-1.jpg](#), [plot-1-3-3-5-2.jpg](#), [plot-1-3-3-5-3.jpg](#)

Case 1-3-3-7: [points-3-3-7.txt](#), [mem-3-3-7.bin](#), [cond-1-3-3-7.txt](#), [sol-1-3-3-7.txt](#), [plot-1-3-3-7-1.jpg](#), [plot-1-3-3-7-2.jpg](#), [plot-1-3-3-7-3.jpg](#)

Case 1-3-3-9: [points-3-3-9.txt](#), [mem-3-3-9.bin](#), [cond-1-3-3-9.txt](#), [sol-1-3-3-9.txt](#), [plot-1-3-3-9-1.jpg](#), [plot-1-3-3-9-2.jpg](#), [plot-1-3-3-9-3.jpg](#)

Case 1-5-5-3: [points-5-5-3.txt](#), [mem-5-5-3.bin](#), [cond-1-5-5-3.txt](#), [sol-1-5-5-3.txt](#), [plot-1-5-5-3-1.jpg](#), [plot-1-5-5-3-2.jpg](#), [plot-1-5-5-3-3.jpg](#)

Case 1-5-5-5: [points-5-5-5.txt](#), [mem-5-5-5.bin](#), [cond-1-5-5-5.txt](#), [sol-1-5-5-5.txt](#), [plot-1-5-5-5-1.jpg](#), [plot-1-5-5-5-2.jpg](#), [plot-1-5-5-5-3.jpg](#)

Case 1-5-5-7: [points-5-5-7.txt](#), [mem-5-5-7.bin](#), [cond-1-5-5-7.txt](#), [sol-1-5-5-7.txt](#), [plot-1-5-5-7-1.jpg](#), [plot-1-5-5-7-2.jpg](#), [plot-1-5-5-7-3.jpg](#)

Case 1-5-5-9: [points-5-5-9.txt](#), [mem-5-5-9.bin](#), [cond-1-5-5-9.txt](#), [sol-1-5-5-9.txt](#), [plot-1-5-5-9-1.jpg](#), [plot-1-5-5-9-2.jpg](#), [plot-1-5-5-9-3.jpg](#)

Case 1-7-7-3: [points-7-7-3.txt](#), [mem-7-7-3.bin](#), [cond-1-7-7-3.txt](#), [sol-1-7-7-3.txt](#), [plot-1-7-7-3-1.jpg](#), [plot-1-7-7-3-2.jpg](#), [plot-1-7-7-3-3.jpg](#)

Case 1-7-7-5: [points-7-7-5.txt](#), [mem-7-7-5.bin](#), [cond-1-7-7-5.txt](#), [sol-1-7-7-5.txt](#), [plot-1-7-7-5-1.jpg](#), [plot-1-7-7-5-2.jpg](#), [plot-1-7-7-5-3.jpg](#)

Case 1-7-7-7: [points-7-7-7.txt](#), [mem-7-7-7.bin](#), [cond-1-7-7-7.txt](#), [sol-1-7-7-7.txt](#), [plot-1-7-7-7-1.jpg](#), [plot-1-7-7-7-2.jpg](#), [plot-1-7-7-7-3.jpg](#)

Case 1-7-7-9: [points-7-7-9.txt](#), [mem-7-7-9.bin](#), [cond-1-7-7-9.txt](#), [sol-1-7-7-9.txt](#), [plot-1-7-7-9-1.jpg](#), [plot-1-7-7-9-2.jpg](#), [plot-1-7-7-9-3.jpg](#)

CASE 2

Conditions: $F_3(\underline{x}) = -F = -0.01$ $\tau_3(x_1, x_2, 0) = 0$. This and (3) imply $\tau_{33}(x_1, x_2, 0) = 0$.

Exact solution:

From previous conditions follows $\tau_{11}(\underline{x}) = \tau_{12}(\underline{x}) = \tau_{13}(\underline{x}) = \tau_{22}(\underline{x}) = \tau_{23}(\underline{x}) = 0$ $\tau_{33}(\underline{x}) = a + b \cdot x_3$. These and the third equation of differential analytical model imply $b = F$. From $\tau_{33}(x_1, x_2, 0) = 0$ and $\tau_{33}(\underline{x}) = a + b \cdot x_3$ follow $a = 0$, and hence is $\tau_{33}(\underline{x}) = F \cdot x_3$. This and (1) imply

$$\varepsilon_{12}(\underline{x}) = \varepsilon_{13}(\underline{x}) = \varepsilon_{23}(\underline{x}) = 0 \quad \varepsilon_{11}(\underline{x}) = -v \cdot F \cdot x_3 / E \quad \varepsilon_{22}(\underline{x}) = -v \cdot F \cdot x_3 / E \quad \varepsilon_{33}(\underline{x}) = F \cdot x_3 / E \quad (12)$$

from which follows (8) (as in CASE 1).

The (5) (10) and (12) imply

$$\{\Theta_1(c) = \Theta_2(c) = 0, \Theta_3(c) = -F \cdot x_3(c) / E; \forall c \in [A, P]\}$$

$$\{\Theta_1(c)=0, \Theta_2(c)=-x_{B3} \cdot v \cdot F/E, \Theta_3(c)=(x_{B2}-x_2(c)) \cdot v \cdot F/E; \forall c \in [P, Q]\}$$

$$\{\Theta_1(c)=-x_{B3} \cdot v \cdot F/E, \Theta_2(c)=0, \Theta_3(c)=(x_{B1}-x_1(c)) \cdot v \cdot F/E; \forall c \in [Q, B]\}$$

From these, (8) and (10) follows

$$\begin{aligned} \Xi_1(\underline{x}) &= v \cdot F \cdot x_3 \cdot (L_1/2 - x_1)/E & \Xi_2(\underline{x}) &= v \cdot F \cdot x_3 \cdot (L_2/2 - x_2)/E \\ \Xi_3(\underline{x}) &= 0.5 \cdot F \cdot (x_3^2 - L_3^2 + v \cdot ((x_1 - L_1/2)^2 + (x_2 - L_2/2)^2))/E \end{aligned} \quad (13)$$

Note: In the following diagrams of this CASE 2, the symbols + (plus), □ (empty square) and ■ (full square) are respectively inherent to \underline{x} , \underline{X} determined by means of $X_i = x_i - \Xi_i$ and (13), and \underline{X} determined by means of $X_i = x_i - \Xi_i$ where Ξ_i is calculated by PEEL.

Case 2-3-3-3: [points-3-3-3.txt](#), [mem-3-3-3.bin](#), [cond-2-3-3-3.txt](#), [sol-2-3-3-3.txt](#), [plot-2-3-3-3-1.jpg](#), [plot-2-3-3-3-2.jpg](#), [plot-2-3-3-3-3.jpg](#)

Case 2-3-3-5: [points-3-3-5.txt](#), [mem-3-3-5.bin](#), [cond-2-3-3-5.txt](#), [sol-2-3-3-5.txt](#), [plot-2-3-3-5-1.jpg](#), [plot-2-3-3-5-2.jpg](#), [plot-2-3-3-5-3.jpg](#)

Case 2-3-3-7: [points-3-3-7.txt](#), [mem-3-3-7.bin](#), [cond-2-3-3-7.txt](#), [sol-2-3-3-7.txt](#), [plot-2-3-3-7-1.jpg](#), [plot-2-3-3-7-2.jpg](#), [plot-2-3-3-7-3.jpg](#)

Case 2-3-3-9: [points-3-3-9.txt](#), [mem-3-3-9.bin](#), [cond-2-3-3-9.txt](#), [sol-2-3-3-9.txt](#), [plot-2-3-3-9-1.jpg](#), [plot-2-3-3-9-2.jpg](#), [plot-2-3-3-9-3.jpg](#)

Case 2-5-5-3: [points-5-5-3.txt](#), [mem-5-5-3.bin](#), [cond-2-5-5-3.txt](#), [sol-2-5-5-3.txt](#), [plot-2-5-5-3-1.jpg](#), [plot-2-5-5-3-2.jpg](#), [plot-2-5-5-3-3.jpg](#)

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Case 2-7-7-3: [points-7-7-3.txt](#), [mem-7-7-3.bin](#), [cond-2-7-7-3.txt](#), [sol-2-7-7-3.txt](#), [plot-2-7-7-3-1.jpg](#), [plot-2-7-7-3-2.jpg](#), [plot-2-7-7-3-3.jpg](#)

Case 2-7-7-5: [points-7-7-5.txt](#), [mem-7-7-5.bin](#), [cond-2-7-7-5.txt](#), [sol-2-7-7-5.txt](#), [plot-2-7-7-5-1.jpg](#), [plot-2-7-7-5-2.jpg](#), [plot-2-7-7-5-3.jpg](#)

Case 2-7-7-7: [points-7-7-7.txt](#), [mem-7-7-7.bin](#), [cond-2-7-7-7.txt](#), [sol-2-7-7-7.txt](#), [plot-2-7-7-7-1.jpg](#), [plot-2-7-7-7-2.jpg](#), [plot-2-7-7-7-3.jpg](#)

Case 2-7-7-9: [points-7-7-9.txt](#), [mem-7-7-9.bin](#), [cond-2-7-7-9.txt](#), [sol-2-7-7-9.txt](#), [plot-2-7-7-9-1.jpg](#), [plot-2-7-7-9-2.jpg](#), [plot-2-7-7-9-3.jpg](#)

Bibliography:

[1] YU. A. AMENZADE, *Theory of Elasticity*, Mir Publishers, 1979, Moscow